Exercise 6.3 (Part 1)

$$\bar{u}^\mu = a^{-1}(1,0,0,0)$$

$$u^\mu = \bar{u}^\mu + \delta u^\mu$$

We have that $g_{\mu\nu} u^\mu u^\nu = -1$ (normalisation condition)

Recall $ds^2 = a^2(\eta)\left[-(1+2A)d\eta^2 + 2B_i dx^i d\eta + (\delta_{ij} + 2E_{ij}) dx^i dx^j\right]$

Then
$$g_{\mu\nu} u^\mu u^\nu = a^2(\eta)(-(1+2A))(a^{-1}+\delta u^0)^2 + 2a^2(\eta)(B_i)(a^{-1}+\delta u^0)(\delta u^i) + (\delta_{ij}+2E_{ij})(\delta u^i)(\delta u^j)a^2(\eta)$$

Discarding 2nd order terms:

$$\begin{aligned} -1 &= a^2(\eta)(-a^{-2}(\eta) - 2Aa^{-2}(\eta) - 2a^{-1}(\eta)\delta u^0) \\ &= -1 - 2A - 2a(\eta)\delta u^0 \end{aligned}$$

$$\Rightarrow \delta u^0 = a^{-1}(-A)$$

$$\begin{aligned} \Rightarrow u^0 &= a^{-1} - a^{-1}A \\ &= a^{-1}(1-A) \end{aligned}$$

Since the background is comoving we can define

$$v_i = \delta u^i a \qquad \left(u^i = \frac{1}{a}\frac{dx^i}{d\eta}\right)$$

Thus,

$$u^\mu = a^{-1}(1-A, v^i)$$

To find u_μ:

$$u_\mu = g_{\mu\nu} u^\nu$$

$$u_0 = g_{0\nu} u^\nu$$

$$= g_{00} u^0 + g_{0i} u^i$$

$$= a^2 (-(1+2A))[a^{-1}(1-A)] + \underbrace{B_i (a^{-1} v_i)}_{2^{nd} \text{ order}}$$

$$= a(-(1+2A)(1-A))$$

$$= -a(1 + A - \cancel{2A^2}) \qquad = a(-(1+A))$$

$$u_i = g_{i\nu} u^\nu = g_{i0} u^0 + g_{ij} u^j$$

$$= B_i a^2 (a^{-1}(1-A)) + a^2 (\delta_{ij} + 2E_{ij})(a^{-1}(v^i))$$

$$= B_i a + a\delta_{ij} v^i \qquad = a(v_i + B_i)$$

$$\Rightarrow u_\mu = a(-(1+A), v_i + B_i)$$

Final results: $u^\mu = a^{-1}(1-A, v^i)$

$$u_\mu = a(-(1+A), v_i + B_i)$$